ELSEVIER

Discrete Mathematics 183 (1998) 81–101

DISCRETE MATHEMATICS

The maximum number of edges in a graph with fixed edge-degree

R.J. Faudree[a], J. Sheehan[b,*]

[a] *Department of Mathematical Sciences, University of Memphis, Memphis, Tennessee 38152, USA*
[b] *Department of Mathemaical Sciences, University of Aberdeen, Dunbar Street, Aberdeen AB9 2TY, UK*

Received 3 October 1995; received in revised form 23 July 1996; accepted 24 February 1997

Abstract

Suppose that $n \geqslant 2t + 2$ $(t \geqslant 17)$. Let G be a graph with n vertices such that its complement is connected and, for all distinct non-adjacent vertices u and v, there are at least t common neighbours. Then we prove that

$$|E(G)| \geqslant \lceil (2t+1)n - 2t^2 - 3)/2 \rceil \quad (n \leqslant 3t - 1)$$

and

$$|E(G)| \geqslant (t+1)n - t^2 - t - 3 \quad (n \geqslant 3t).$$

Furthermore, the results are sharp.

1. Introduction

All graphs are both finite and simple. Let G be a graph with vertex set $V(G)$ and edge set $E(G)$. Set $q(G) = |E(G)|$. Let $v \in V(G)$. Then $\deg_G(v)$ denotes the degree of v in G (as usual the suffix is used throughout only if there is a danger of ambiguity) and $N(v)$ denotes the set of neighbours of v in G. Let $\delta(G)$ denote the minimum degree of G. The complement of G is denoted by $\bar{G}$. When $X \subseteq V(G)$, $\langle X \rangle$ denotes the subgraph of G induced by X.

Let x be a real number. Then $\lceil x \rceil$ and $\lfloor x \rfloor$ denote, respectively, the upper and lower integer part of x. All this notation is fairly standard. The main definitions (and notation) particular to this paper now follow. Set

$$\text{pos}\langle x \rangle = \max\{0, x\}.$$

* Corresponding author. E-mail: john_sheehan@aberdeen.ac.uk.

PII S0012-365X(97)00078-2

Suppose that $u, v \in V(G)$ $(u \neq v)$. Set

$$d(uv) = |N(u) \cap N(v)| \quad \text{if } uv \in E(\bar{G}),$$

$$d^*(uv) = |N(u) \cup N(v)| \quad \text{if } uv \in E(G).$$

A graph G is said to have the *k-degree property* if G is connected and

$$d^*(uv) \leqslant k \quad (\forall uv \in E(G)).$$

A graph G is said to have the *k-parent property* if $\bar{G}$ is connected and

$$d(uv) \geqslant k \quad (\forall uv \in E(\bar{G})).$$

Let $\mathscr{D}(k)$ *and* $\mathscr{P}(k)$ be, respectively, the sets of graphs with the k-degree and k-parent property. Clearly, if G has n vertices then

$$G \in \mathscr{D}(k) \quad \text{if and only if } \bar{G} \in \mathscr{P}(n-k). \tag{1}$$

Set

$$\mathscr{D}(n, k) = \{G \in \mathscr{D}(k): |V(G)| = n\}$$

(then $\mathscr{P}(n, k)$ has the obvious meaning).

The k-degree property arose naturally when one of the authors was considering extremal problems involving weakly k-linked graphs (see, for example, [3]). It is also a property which arises in, for example, [1; 2, p. 16, Theorem 2.3.2].

Sometimes, for example, for small fixed values of k, it is easier to work with $\mathscr{D}(k)$, rather than $\mathscr{P}(k)$ and vice versa.

Set

$$f(n, k) = \min\{q(G): G \in \mathscr{P}(n, k)\} \quad (n \geqslant k + 4)$$

and

$$g(n, k) = \max\{q(G): G \in \mathscr{D}(n, k)\} \quad (k \geqslant 4).$$

Then, from (1)

$$f(n, k) = \binom{n}{2} - g(n, n-k). \tag{2}$$

2. Examples

(i) It is easy to show that $g(n, 4) = n$ and the unique extremal graph is $C_n (n \geqslant 4)$.

(ii) An induction argument shows that $g(5k, 5) = 7k$ and the unique extremal graph $G(k)$ $(k \geqslant 2)$ is shown in Fig. 1.

Let n, k $(\geqslant 4)$ be integers with $2n \geqslant 2k \geqslant n + 2$. Let $H_1 = H_1(n, k)$ be the graph described as follows.

Write $\theta = \lceil (2k - n - 1)/2 \rceil$ and set

$$K^*(k-1) = K(k-1) - \theta K(2),$$

i.e. $K^*(k-1)$ is the complete graph $K(k-1)$ with a matching of size θ deleted. Then H_1 (see Fig. 2) consists of the disjoint union $K^*(k-1) \cup K(n-k)$ together with a vertex v of degree $2k-n$; v is adjacent to exactly one vertex of $K(n-k)$ and to $2k-n-1$ of the vertices of $K^*(k-1)$ which have degree $k-3$ in $K^*(k-1)$.

Write $t=n-k$. Then set $G_1(n,t)=\bar{H}_1$ $(n \geqslant 2t+2, t \geqslant 2)$.

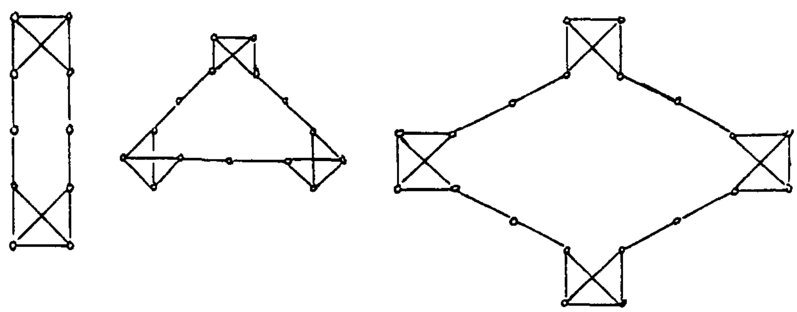
Let n, k be integers with $3k \geqslant 2n \geqslant 2k+4$. Let $H_2 = H_2(n,k)$ be the graph described as follows.

The graph H_2 (see Fig. 3) consists of the disjoint union $K(k-2) \cup K(n-k)$ together with two distinct vertices v_1 and v_2 whose adjacencies we now describe. Let $V(K(k-2)) = V_1 \cup V_2$ where $|V_1| = \lfloor (k-2)/2 \rfloor$ and $|V_2| = \lceil (k-2)/2 \rceil$. Then v_i is adjacent to the vertices of V_i and to one vertex u_i of K_{n-k} $(i=1,2;\ u_1 \neq u_2)$.

Write $t=n-k$ and set $G_2(n,t)=\bar{H}_2$ $(n \geqslant 3t, t \geqslant 2)$.

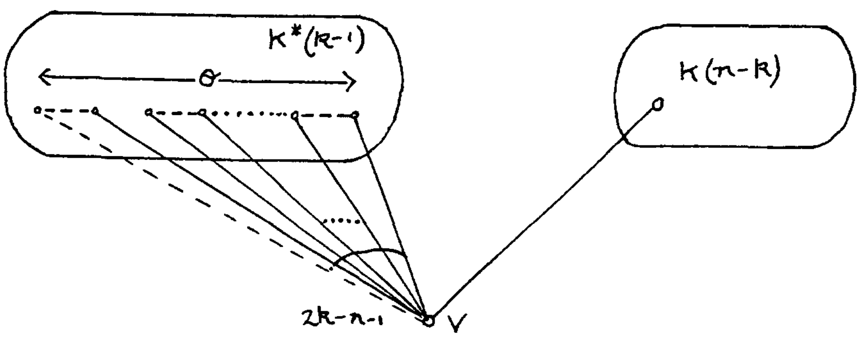
Write

$$f^*(n,t) = \begin{cases} \lceil ((2t+1)n - 2t^2 - 3)/2 \rceil & (2t+2 \leqslant n \leqslant 3t-1,\ t \geqslant 2), \\ (t+1)n - t^2 - t - 3 & (n \geqslant 3t, t \geqslant 2). \end{cases}$$



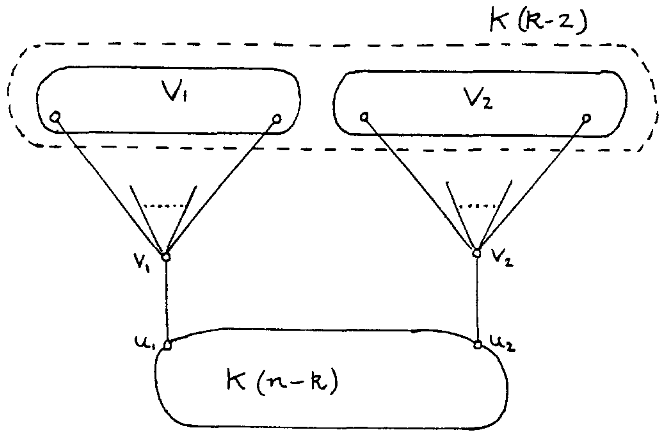
G(2) G(3) G(4)

Fig. 1.



$H_1(n,k)$

Fig. 2.



$H_2(n,k)$

Fig. 3.

Remark. Notice that for $n \geqslant 2t + 3$

$$\lceil((2t+1)n - 2t^2 - 3)/2\rceil \leqslant (t+1)n - t^2 - t - 3.$$

When $n = 2t + 2$ the inequality is reversed with the two sides differing by one.

Lemma 1. *Suppose that* $n \geqslant 2t + 2$ $(t \geqslant 2)$.
Then $f(n, t) \leqslant f^*(n, t)$.

Proof. It is easy to check that $G_i(n, t) \in \mathscr{P}(n, t)$ $(i = 1, 2)$, that $q(G_1(n, t)) = f^*(n, t)$ $(2t + 2 \leqslant n \leqslant 3t - 1)$ and $q(G_2(n, t)) = f^*(n, t)$ $(n \geqslant 3t)$. $\square$

The rest of this paper consists of a proof that equality holds in Lemma 1 $(t \geqslant 17)$. We proceed by contradiction.

Thus assume that $G \in \mathscr{P}(n, t)$ $(n \geqslant 2t + 2,\ t \geqslant 1)$ and $q(G) < f^*(n, t)$. Then the crucial notation is as follows.

3. Notation

Set $\delta(G) = t + s$ $(s \geqslant 1)$ where $\delta(G)$ is the minimum degree of G. Notice that since $\bar{G}$ is connected s is at least 1. Choose $v \in V(G)$ so that $\deg(v) = t + s$. Set $X = N(v)$ and $Y = V(G) \backslash (X \cup \{v\})$. Set $m = |Y| = n - s - t - 1$ (see Fig. 4).

Lemma 2. (i) $(t + s)n \leqslant 2(f^*(n, t) - 1)$.
(ii) $t \geqslant 3s - 2$ $(2t + 2 \leqslant n \leqslant 3t - s + 1)$.

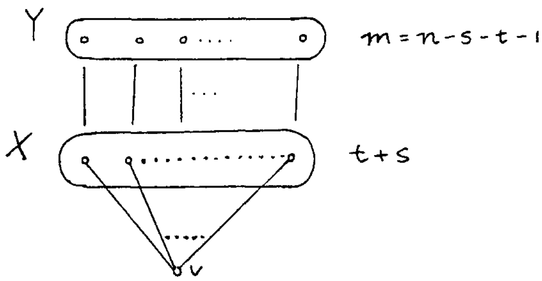


Fig. 4.

Proof. (i) By definition

$$(t+s)n \leqslant 2q(G) \leqslant 2(f^*(n,t)-1). \tag{3}$$

(ii) From (3) and the definition of $f^*(n,t)$,

$$s \leqslant (t+1) - (2t^2+4)/n. \tag{4}$$

Write $t = \varepsilon n$. Then since $n \leqslant 3t$, $\varepsilon \geqslant \frac{1}{3}$ and from (4)

$$s \leqslant t/3 + (1 - 4/n)$$

and

$$t \geqslant 3s - 2. \quad \square$$

4. Notation

Let X, $Y \subseteq V(G)$. Write

$$q(X, Y) = |\{xy \in E(G) \colon x \in X, y \in Y\}|.$$

For brevity set $q(X) = q(X, X)$ and $q(G) = q(V(G)) = |E(G)|$. Using licence with parentheses, $q(x, y) = 0$ or 1 according as x is or is not adjacent to y. Again $q(x, V(G)) = |N(x)| = \deg(x)$.

5. Assumptions for Lemma 3 below

Assume that α ($\geqslant 0$) and β are integers such that $\alpha + \beta \geqslant s \geqslant 1$ and $t \geqslant s \geqslant \beta > 0$. Let $m^* \in \{m, m-1\}$. Let

$$\lambda(m^*, \alpha, \beta) = \alpha + 2\beta + \operatorname{pos}\left\langle \frac{2(s-\beta)(t-\alpha)}{m^*} \right\rangle.$$

Finally, suppose that β_0 ($\leqslant s-1$) is a positive integer such that $\beta \geqslant \beta_0$. Subject to these constraints let $\lambda^*(m^*, \beta_0)$ be the smallest value of $\lambda(m^*, \alpha, \beta)$.

Lemma 3. (i) $\lambda^*(m^*, \beta_0) \in \{\lambda(m^*, 0, s), \lambda(m^*, s-\beta_0, \beta_0)\}$ $(n \geqslant 2t+2,\ t \geqslant 4)$.
(ii) $\lambda^*(m, 0) = \lambda(m, s, 0)$ $(n \geqslant 3t - s + 1,\ n \geqslant 2t+2,\ t \geqslant 4)$.

Proof. We wish to minimize $\lambda(m^*, \alpha, \beta)$ in the region (shaded) of Fig. 5.

We first show that λ is monotone increasing for fixed α. This is trivially true if either $\alpha \geqslant t$ or if $\alpha = 0$ (in which case $\beta = s$). So suppose that α is fixed and $0 < \alpha < t$. Then

$$\lambda(m^*, \alpha, \beta+1) - \lambda(m^*, \alpha, \beta) = \frac{2}{m^*}(m^* - t + \alpha) \geqslant 0 \tag{5}$$

if $n \geqslant 2t + s + \delta(m^*)$ where $\delta(m^*) = 1$ if $m^* = m - 1$ and $\delta(m^*) = 0$ if $m^* = m$.

So assume that

$$n \leqslant 2t + s + \delta(m^*) - 1. \tag{6}$$

Then, from Lemma 2(i) and (6),

$$(n-t)n \leqslant 2((t+1)n - t^2 - t - 3),$$

that is,

$$n^2 - 3tn + 2t^2 + 2t + 6 \leqslant 0$$

which, since $n \geqslant 2(t+1)$, is false. So $n \geqslant 2t + s + \delta(m^*)$ and hence

$$\lambda(m^*, \alpha, \beta+1) \geqslant \lambda(m^*, \alpha, \beta) \tag{7}$$

in all cases.

Now assume that $\beta = \beta_0$. Trivially, if $\alpha \geqslant t$ then λ is monotone increasing. So suppose that $s - \beta_0 \leqslant \alpha < t$. We have

$$\lambda(m^*, \alpha+1, \beta_0) - \lambda(m^*, \alpha, \beta_0) = 1 - \frac{2(s-\beta_0)}{m^*} \geqslant 0 \tag{8}$$

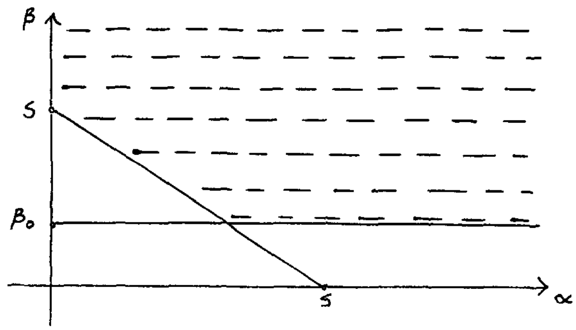


Fig. 5.

if and only if $n \geqslant 3s + t + 1 - 2\beta_0 + \delta(m^*)$. Now assume that

$$n \leqslant 3s + t + \delta(m^*). \tag{9}$$

Then, from Lemma 2(i) and (9)

$$(n + 2t - 1)n \leqslant 6((t + 1)n - t^2 - t - 3),$$

that is,

$$n^2 - (4t + 7)n + 6(t^2 + t + 3) \leqslant 0$$

which, since $n \geqslant 2(t + 1)$, is false. Hence, $n \geqslant 3s + t + 1 + \delta(m^*) \geqslant 3s + t + 1 - 2\beta_0 + \delta(m^*)$ and in all cases

$$\lambda(m^*, \alpha + 1, \beta_0) \geqslant \lambda(m^*, \alpha, \beta_0). \tag{10}$$

From (7) and (10), $\lambda^*(m^*, \beta_0) = \lambda(m^*, s - \beta, \beta)$ for some $\beta(s \geqslant \beta \geqslant \beta_0,\ s - 1 \geqslant \beta_0)$. But $\lambda(m^*, s - \beta, \beta)$ is a quadratic in β and has its minimum value at either $\beta = \beta_0$ or $\beta = s$.

Finally, $\lambda(m, 0, s) \geqslant \lambda(m, s, 0)$ if and only if $n \geqslant 3t - s + 1$. $\square$

The relevance of Lemma 3 becomes clearer with the introduction of yet more notation. Suppose that $y \in Y$. Set $\alpha(y) = q(y, Y)$ and $\beta(y) = q(y, X) - t$ (see Fig. 6). Since $d(vy) \geqslant t$, $\beta(y) \geqslant 0$ and since $\delta(G) = t + s$, $\alpha(y) + \beta(y) \geqslant s$, $\beta(y) \leqslant s$. Finally, set

$$\beta_0 = \min\{\beta(y) \colon y \in Y\}.$$

Since $\bar{G}$ is connected, $\beta_0 \leqslant s - 1$.

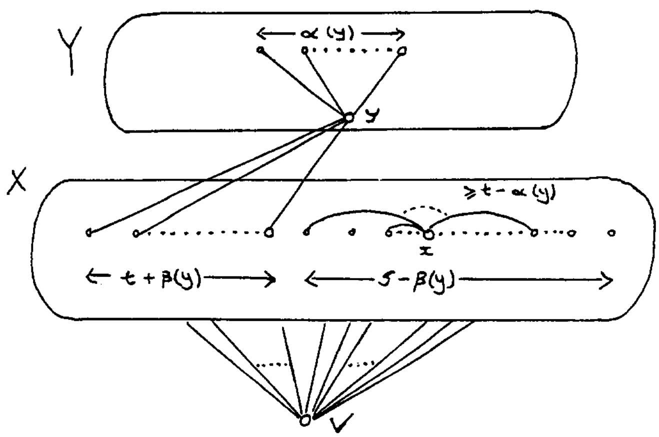


Fig. 6.

6. Comment

Recall that if $y \in Y$ then

$$\lambda(m^*, \alpha(y), \beta(y)) = \alpha(y) + 2\beta(y) + \mathrm{pos}\left\langle \frac{2((s-\beta(y))(t-\alpha(y))}{m^*} \right\rangle, \tag{11}$$

where $m^* \in \{m, m-1\}$.

We interpret λ as follows. Suppose that $m^* = m$. There exist $s - \beta(y)$ vertices x in X not adjacent to y; since $d(xy) \geqslant t$ for each such x, $q(x, X) \geqslant t - \alpha(y)$ (see Fig. 6).

Now each $y \in Y$ is possibly non-adjacent to the same vertices of X. With the weighting given in (11), $\lambda(m^*, \alpha(y), \beta(y)) + 2t$ may be interpreted as the *minimum contribution* of y towards the double edge count $2q(G)$ (recall that $q(y, X) = t + \beta(y)$, which explains the additional $2t$).

Finally write $\lambda(Y)$ for the totality of the contributions of the m elements of Y towards the double edge-count $2q(G)$.

If $Y' \subseteq Y$ and $|Y'| = m - 1$ then $\lambda(Y')$ has the analogous meaning with $m^* = m - 1$.

Lemma 4.
(i) $2q(G) \geqslant 2(t+s) + m(2t+s) + 2s(t-s)$ $(n \geqslant 3t - s + 1,\ n \geqslant 2t + 2,\ s \geqslant 2)$.
(ii) *Write* $n = 3t - s - \theta$ $(t - s - 2 \geqslant \theta \geqslant -1,\ s \geqslant 2)$. *Then*

$$2q(G) \geqslant m(2s - 1 + 2t) + 5t + 2s - 4 - \mathrm{pos}\langle 4s - 2 - (\theta+1)(s-1)\rangle.$$

Proof. (i) By Lemma 3(ii) and the definition of $\lambda(Y)$

$$\lambda(Y) \geqslant 2mt + m\lambda(m, s, 0). \tag{12}$$

Hence,

$$\begin{aligned} 2q(G) &\geqslant 2q(v, X) + \lambda(Y) \\ &\geqslant 2(t+s) + m(2t+s) + 2s(t-s). \end{aligned}$$

(ii) The proof of Lemma 4(ii) is given in the appendix. □

Lemma 5. *Suppose that* $G \in \mathscr{P}(n, t)$ $(n \geqslant 2t + 2,\ t \geqslant 17)$. *Then* $q(G) \geqslant f^*(n, t)$ *if either*

(i) $n \geqslant 3t - s + 1, \quad s \geqslant 3$

or

(ii) $n \leqslant 3t - s + 1, \quad s \geqslant 2.$

Proof. Assume that

$$q(G) \leqslant f^*(n, t) - 1. \tag{13}$$

The edges of G are counted in (14) and (15) in two different ways. Using (13) and recalling that $\delta(G) = t + s$,

$$(t+s)^2 - q(X) \leqslant q(G) \leqslant f^*(n,t) - 1 \tag{14}$$

$$m\left(t + \frac{s}{2}\right) + (t+s) + q(X) \leqslant q(G) \leqslant f^*(n,t) - 1. \tag{15}$$

From (14) and (15)

$$4f^*(n,t) \geqslant (s+2t)n + s^2 + (t+1)s + 4. \tag{16}$$

Case (i): *Suppose* $n \geqslant 3t - s + 1$, *where* $s \geqslant 3$. From Lemma (4)(i), (13) and the definition of $f^*(n,t)$ (and specifically the remark following the definition)

$$(s-2)n \leqslant (s-2)t + 3s^2 - s - 8 \tag{17}$$

for $n \geqslant 2t + 3$; if $n = 2t + 2$ then $s = t - 1$ which from Lemma 2(ii) is impossible. From (17)

$$n \leqslant t + 3s + 5 + \left\lfloor \frac{2}{s-2} \right\rfloor. \tag{18}$$

From (16) and the definition of $f^*(n,t)$,

$$(2t - s + 4)n \geqslant 4t^2 + s^2 + st + 4t + s + 16. \tag{19}$$

Hence, from (18), (19) and since $s \geqslant 3$,

$$2s^2 - 2(t+1)s + t^2 - 7t - 6 \leqslant 0$$

which is impossible since $t \geqslant 17$.

Case (ii): *Suppose* $n \leqslant 3t - s + 1$, *where* $s \geqslant 2$. From Lemma 4(ii), (13) and the definition of $f^*(n,t)$

$$2(s-1)n \leqslant 4(s-1)t + 2s^2 - s - 1 + \operatorname{pos}\langle 4s - 2 - (\theta+1)(s-1)\rangle \tag{20}$$

where $n = 3t - s - \theta$ $(t - s - 2 \geqslant \theta \geqslant -1)$.

Assume that either $3 \geqslant \theta \geqslant 0$ or $\theta = 4$ and $s = 2$. Then, from (20),

$$2(s-1)(3t - s - \theta) \leqslant 4(s-1)t + (s-1)(2s+5) - (\theta+1)(s-1) + 2$$

i.e.

$$2t \leqslant 4s + 4 + \theta + \frac{2}{(s-1)} \tag{21}$$

From Lemma 2(ii) and (21)

$$6s - 4 \leqslant 4s + 4 + \theta + \frac{2}{s-1}. \tag{22}$$

Since $t \geqslant 17$, (21) and (22) yield a contradiction. So we may now suppose that $\theta \geqslant 4$ and if $\theta = 4$ then $s \geqslant 3$. From (20),

$$2(s-1)n \leqslant 4(s-1)t + (s-1)(2s+1). \tag{23}$$

Hence,

$$n \leqslant 2t + s. \tag{24}$$

From (16) and the definition of $f^*(n, t)$

$$(2t - s + 2)n \geqslant 4t^2 + s^2 + st + s + 8. \tag{25}$$

From (24) and (25)

$$0 \geqslant 2s^2 + (t-1)s - 4t + 8. \tag{26}$$

Hence from (26), $2 \leqslant s \leqslant 3$. From (24) when $s = 2$, $n = 2t + 2$ and when $s = 3$, $n \leqslant 2t + 3$. Now Lemma 2(i) yields a contradiction in each case. □

Lemma 6. *Suppose that* $G \in \mathscr{P}(n, t)$ *where* $n \geqslant 2t + 2$ $(t \geqslant 17)$ *and* $s = 1$, *then* $q(G) \geqslant f^*(n, t)$.

Proof. Since $s = 1$ (see Fig. 4), $|X| = t + 1$, $m = |Y| = n - t - 2$. Let $y \in Y$. Then, since $d(vy) \geqslant t$, $q(y, X) \in \{t, t+1\}$. Set

$$Y_0 = \{y \in Y: q(y, X) = t + 1\}.$$

Since $\bar{G}$ is connected $Y_0 \neq Y$. Set

$$X^* = \{x \in X: q(x, Y) < m\}.$$

Since $Y_0 \neq Y$, $X^* \neq \emptyset$. Suppose that $X^* = \{x_i: i = 1, 2, \ldots, r\}$ $(r \geqslant 1)$. Set

$$Y_i = \{y \in Y: q(x_i, y) = 0\} \quad (i = 1, 2, \ldots, r).$$

Then (see Fig. 7), $(Y_0, Y_1, \ldots, Y_r)$ is a partition of Y. Write

$$Y^* = \bigcup_{i=1}^{r} Y_i.$$

We begin by proving two claims.

Claim 1. *Let* $y_i \in Y_i$, $y_j \in Y_j$ $(i > j > 0)$ *and suppose that* $q(y_i, y_j) = 0$. *Then there exists* $y \in Y$ *with* $q(y, y_i) = q(y, y_j) = 1$.

Proof (Claim 1). This follows since $|N(y_i) \cap N(y_j) \cap X| = t - 1$ and $d(y_i y_j) \geqslant t$. □

Claim 2. *Suppose that* $r \geqslant 2$. *Then*
(i) *if* $\langle Y^* \rangle$ *is disconnected,* $q(Y) \geqslant |Y^*|$,
(ii) *if* $\langle Y^* \rangle$ *is connected,* $q(Y) \geqslant |Y^*| - 1$.

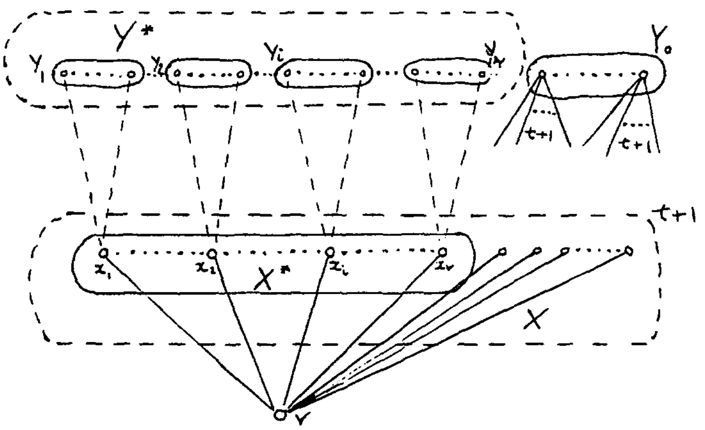


Fig. 7. See proof of Lemma 6.

Proof (Claim 2). Let $K_1, K_2, \ldots, K_a$ $(a \geqslant 1)$ be the components (see Fig. 7) of $\langle Y^* \rangle$.

Firstly, suppose that $a \geqslant 2$. Suppose that for some i $(1 \leqslant i \leqslant a)$, $q(K_i, Y_0) = 0$. Set $K = K_i$. Choose any $y \in V(K)$. Then, using Claim 1, for all $y^* \in Y^* \setminus V(K)$, $\{y^*, y\} \subseteq Y_j$ for some fixed $j (1 \leqslant j \leqslant r)$. Remembering that y is an arbitrary element of $V(K)$ this implies that $Y^* \subseteq Y_j$. Hence $r = 1$ which is not so. Therefore, $q(K_i, Y_0) \geqslant 1$ $(1 \leqslant i \leqslant a)$. Hence, $q(Y) \geqslant (|Y^*| - a) + a = |Y^*|$. Finally, if $a = 1$ then $\langle Y^* \rangle$ is connected and $q(Y) \geqslant q(Y^*) \geqslant |Y^*| - 1$. □

Proof of Lemma 6 (*continued*). Set

$$\alpha = \min\{q(y, Y): y \in Y^*\}. \tag{27}$$

Since $\delta(G) = t + 1$ and $Y^* \neq \emptyset$, $\alpha \geqslant 1$.

Case (i): *Suppose* $r \geqslant 2$. From Claim 2 and (27)

$$q(Y) \geqslant \max\left\{(l - 1), (l\alpha)/2, l\alpha - \binom{l}{2}\right\}, \tag{28}$$

where $l = |Y^*|$.

Choose $y_1 \in Y^*$ with $q(y_1, Y) = \alpha$. Without loss of generality suppose that $y_1 \in Y_1$. Then, since $d(x_1 y_1) \geqslant t$,

$$q(x_1, X) \geqslant t - \alpha. \tag{29}$$

Always

$$q(G) = q(v, X) + q(X, Y) + q(X) + q(Y) \tag{30}$$

$$\geqslant (t + 1) + ((n - t - 2)t + |Y_0|) + (t - \alpha) + q(Y). \tag{31}$$

First suppose that $n \geqslant 2t+3$. Then, from (28), (31) and the definition of $f^*(n,t)$ (specifically the remark following the definition)

$$q(G) \geqslant (t+1) + ((n-t-2)t + |Y_0|) + (t-\alpha) + (l\alpha)/2 \tag{32}$$

$$\geqslant f^*(n,t) + ((l-2)(\alpha-2))/2. \tag{33}$$

Since $r \geqslant 2$, $l \geqslant 2$. Hence if $\alpha \geqslant 2$, $q(G) \geqslant f^*(n,t)$.

On the other hand, if $\alpha = 1$ from (28) (using $l-1$ as the maximum) and (31)

$$q(G) \geqslant (n-t-1)(t+1) - l + (t-1) + l - 1$$

$$\geqslant f^*(n,t). \tag{34}$$

Now, suppose throughout the remainder of the proof that $n = 2t+2$. Suppose $\alpha \geqslant 3$. Then $l = 2$ otherwise using (32) we are done. So from (28) (using $l\alpha - \binom{l}{2}$ as the maximum) and (31)

$$q(G) \geqslant (t+1)^2 - l + (t-\alpha) + l\alpha - \binom{l}{2}$$

$$\geqslant t^2 + 3t = f^*(2(t+1), t).$$

Suppose that $\alpha = 2$ and $q(G) \leqslant t^2 + 3t - 1$. Then (32) becomes an equality, i.e. for each $y \in Y$, $q(y, Y) = 2$ and from (29), $q(X) = t-2$. Since $r \geqslant 2$, we may assume $Y_2 \neq \phi$. Choose $y_2 \in Y_2$ then $q(y_2, Y) = 2$ and hence since $q(x_2, y_2) = 0$ and $d(x_2 y_2) \geqslant t$, $q(x_2, X) \geqslant t-2$. Hence $q(X) \geqslant 2(t-2) - 1 > t-2$ (since $t > 3$) which is a contradiction. So again $q(G) \geqslant f^*(2(t+1), t)$.

Now suppose that $\alpha = 1$ and $q(G) \leqslant t^2 + 3t - 1$. Then (34) becomes an equality, i.e. from (29), $q(x_1, X) = q(X) = t-1$, $q(Y) = l-1$ and hence from Claim 2, $\langle Y^* \rangle$ is a tree with $q(Y) = q(Y^*)$. Since $r \geqslant 2$, we may assume that $Y^* \backslash Y_1 \neq \emptyset$. Choose $y \in Y^* \backslash Y_1$ and set $q(y, Y) = t-k$ (since $|Y| = t$, $k \geqslant 1$). Without loss of generality assume that $y \in Y_2$. Then since $q(x_2, y) = 0$, $d(x_2 y) \geqslant t$ and $q(x_2, X) \geqslant k$. It follows that since $q(x_1, X) = q(X) = t-1$, $q(x_1, x_2) = 1$, $k = 1$ and $q(y, Y) = t-1$. Since $\langle Y^* \rangle$ is a tree this implies that $|Y^* \backslash Y_1| = 1$ and indeed since $q(Y) = q(Y^*)$, $|Y \backslash Y_1| = 1$. Without loss of generality, let $Y_2 = \{y_2\}$ where $q(y_2, Y) = t-1$. Since $\langle Y^* \rangle$ is a tree, $q(y, Y) = 1$ for all $y \in Y (y \neq y_2)$ (i.e. $\langle Y \rangle \cong K_{1,t-1}$). Since $q(x_1, X) = t-1$ there exists $x \in X$, $(x \neq x_2)$, $q(x, x_1) = 0$. But since $q(X) = t-1$, $q(x, X) = 0$. Hence, since $d(xx_1) \geqslant t$, $q(x_1, Y) = t-1$. Hence $|Y_1| = 1$ and so $|Y_1| = |Y_2| = 1$ and $|Y| = 2$ which is impossible since $|Y| = t > 2$.

Case (ii): *Suppose* $r = 1$. Set $q(x_1, X) = t-k$ $(0 \leqslant k \leqslant t)$. Notice that since $r = 1$, $k \geqslant 1$ otherwise $\bar{G}$ is disconnected. For each $y \in Y_1$, since $d(yx_1) \geqslant t$, $q(y, Y_0) \geqslant k$. Therefore, from (30),

$$q(G) \geqslant (t+1) + (n-t-2)(t+1) - l + (t-k) + lk. \tag{35}$$

Hence if $n \geqslant 2t + 3$, from (35),

$$q(G) \geqslant f^*(n, t) + (l(k-1) - (k-2))$$
$$\geqslant f^*(n, t).$$

Finally, when $n = 2t + 2$, since $l \geqslant 1$,

$$q(G) \geqslant f^*(2(t+1), t) + l(k-1) - (k-1)$$
$$\geqslant f^*(2(t+1), t). \qquad \square$$

Lemma 7. *Suppose that $G \in \mathscr{P}(n, t)$ where $n \geqslant 3t (t \geqslant 17)$ and $s = 2$, then $q(G) \geqslant f^*(n, t)$.*

Proof. Recall that for $n \geqslant 3t$ and $s = 2$,

$$f^*(n, t) = (t+1)n - t^2 - t - 3 \tag{36}$$

$|X| = t + 2, m = |Y| = n - t - 3$. For $y \in Y$,

$$\alpha(y) + \beta(y) \geqslant 2, \quad \beta(y) \leqslant 2. \tag{37}$$

Also, by definition

$$q(X, Y) = (n - t - 3)t + \sum_{y \in Y} \beta(y) \tag{38}$$

and

$$q(Y) = \frac{1}{2} \sum_{y \in Y} \alpha(y). \tag{39}$$

Set

$$w(y) = \tfrac{1}{2}\alpha(y) + \beta(y); \qquad w(Y) = \sum_{y \in Y} w(y).$$

Then, from (38) and (39),

$$q(G) = q(v, X) + q(X, Y) + q(Y) + q(X)$$
$$= (t+2) + (n - t - 3)t + w(Y) + q(X). \tag{40}$$

Vertex $y \in Y$ is said to be an $(\geqslant \alpha, \geqslant \beta)$-vertex if $\alpha(y) \geqslant \alpha$, $\beta(y) \geqslant \beta$ (variations on this notation will have their obvious meanings). Set

$$\mathscr{B}_i = \{y \in Y \colon \beta(y) = i\}$$

and

$$b_i = |\mathscr{B}_i| \quad (i = 0, 1, 2).$$

Since $\bar{G}$ is connected

$$b_2 \leqslant m - 1 = n - t - 4. \tag{41}$$

From (36) and (40), $q(G) \geqslant f^*(n, t)$ if and only if

$$w(Y) + q(X) \geqslant n + t - 5. \tag{42}$$

Since, from (37) $w(Y) \geqslant n - t - 3$ the "edge deficit" is $2t - 2$.

Choose $y_0 \in Y$ so that $\beta(y_0) \leqslant 1$ (from (41) such a y exists) and choose $x \in X$ with $q(x, y_0) = 0$. Since $d(xy_0) \geqslant t$

$$q(X) \geqslant q(x, X) \geqslant t - \alpha(y_0). \tag{43}$$

We begin by proving three claims.

Claim 1. *Suppose that, for all $y \in \mathscr{B}_i$, $\alpha(y) \geqslant 2(2 - i)$ $(i = 0, 1)$. Then $q(G) \geqslant f^*(n, t)$.*

Proof (Claim 1). Refine the choice of $y_0 \in \mathscr{B}_0 \cup \mathscr{B}_1$ so that $\alpha(y_0)$ attains the

$$\min\{\alpha(y) - 2(2 - i)\colon y \in \mathscr{B}_i,\ i = 0, 1\}.$$

Then, since for all $y \in \mathscr{B}_i$, $\alpha(y) \geqslant 2(2 - i)$, from (37), $w(y) \geqslant 2$ for all $y \in Y$ and

$$w(Y) \geqslant 2(n - t - 3) + \tfrac{1}{2}(b_1 + b_2)(\alpha(y_0) - 2(2 - i)),$$

where $y_0 \in \mathscr{B}_i$, $i = 0, 1$. If $b_1 + b_2 = 1$ then

$$w(Y) \geqslant 2(n - t - 3) + (\alpha(y_0) - 2(2 - i)).$$

So in all cases

$$w(Y) \geqslant 2(n - t - 3) + \alpha(y_0) - 2(2 - i). \tag{44}$$

Therefore from (42)–(44) $q(G) \geqslant f^*(n, t)$ (recall here and below that by assumption $n \geqslant 3t, t \geqslant 17$). $\square$

Claim 2. *Suppose that for all $y \in \mathscr{B}_i$, $\alpha(y) \geqslant 3 - i$ $(i = 0, 1)$ and $\alpha(y^*) = 3$ for some $y^* \in \mathscr{B}_0$. Then $q(G) \geqslant f^*(n, t)$.*

Proof (Claim 2). Since $y^* \in \mathscr{B}_0$ there exist $x_1, x_2 \in X$ $(x_1 \neq x_2)$ such that $q(y^*, x_i) = 0$ $(i = 1, 2)$. Hence, since $d(x_i y^*) \geqslant t$,

$$q(X) \geqslant q(x_1, X) + q(x_2, X) \geqslant 2(t - 3). \tag{45}$$

Since $\alpha(y) \geqslant 3 - i$, $w(y) \geqslant 3/2$ for all $y \in Y$. Hence,

$$w(Y) \geqslant 3(n - t - 3)/2. \tag{46}$$

From (42), (45), (46), $q(G) \geqslant f^*(n, t)$. $\square$

Claim 3. *Suppose that $\alpha(y) \geqslant 3$ for all $y \in \mathscr{B}_0$ ($\mathscr{B}_0$ is possibly empty). Then $q(G) \geqslant f^*(n, t)$.*

Proof (Claim 3). From Claims 1 and 2 we may assume that there exists $y \in \mathscr{B}_1$ with $\alpha(y) = 1$ (recall, using (37), that $\alpha(y) \geqslant 1$ for $y \in \mathscr{B}_1$), i.e. y is an $(=1, =1)$-vertex.

Suppose that $y_1, y_2 \in Y$ ($y_1 \neq y_2$) are $(=1, =1)$-vertices. Then y_1 and y_2 have the same neighbourhoods in X. Suppose otherwise. Then there exist $x_1, x_2 \in X$ ($x_1 \neq x_2$) such that $q(x_i, y_i) = 0$ ($i = 1, 2$). Then since $d(x_i y_i) \geqslant t$

$$\begin{aligned} q(X) &\geqslant q(x_1, X) + q(x_2, X) - 1 \\ &\geqslant 2(t-1) - 1. \end{aligned} \tag{47}$$

Again since $w(y) \geqslant \frac{3}{2}$ for all $y \in Y$,

$$w(Y) \geqslant 3(n - t - 3)/2. \tag{48}$$

From (42), (47) and (48), $q(G) \geqslant f^*(n, t)$.

So we may now assume that all $(=1, =1)$-vertices have the same neighborhood in X. Let $x \in X$ be the vertex which is adjacent to no $(1, 1)$-vertex.

Assume that $\mathscr{B}_0 \neq \emptyset$. Choose $y^* \in \mathscr{B}_0$ and $x^* \in X$ such that $x^* \neq x$ and $q(x^*, y^*) = 0$ (see Fig. 8).

Furthermore, choose y^* so that

$$\alpha(y^*) = \min\{\alpha(y)\colon y \in \mathscr{B}_0\}.$$

Since $d(x^* y^*) \geqslant t$ and if $\alpha(y^*) \leqslant t - 1$,

$$\begin{aligned} q(X) &\geqslant q(x, X) + q(x^*, X) - 1 \\ &\geqslant (t-1) + (t - \alpha(y^*)) - 1. \end{aligned} \tag{49}$$

If $\alpha(y^*) \geqslant t$ then

$$q(X) \geqslant t - 1. \tag{50}$$

Since $\alpha(y) \geqslant 3$ for all y in $\mathscr{B}_0$,

$$w(Y) \geqslant 3(n - t - 3)/2 + b_0(\alpha(y^*) - 3)/2. \tag{51}$$

Hence, from (42), (49)–(51), $q(G) \geqslant f^*(n, t)$.

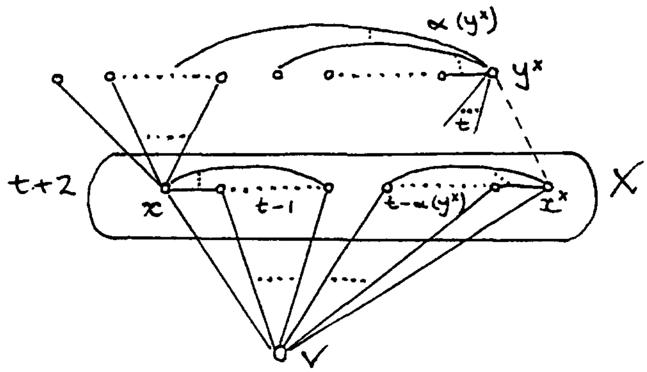


Fig. 8.

So we may now suppoe that $\mathscr{B}_0 = \emptyset$. Set

$$a = |\{y \in Y: \alpha(y) = \beta(y) = 1\}| \quad (a \geqslant 1)$$

$$b = |\{y \in Y: \alpha(y) \geqslant 2, \beta(y) = 1\}|$$

$$c = |\{y \in Y: \beta(y) = 2\}|.$$

Then $a + b + c = m = n - t - 3$ (in the earlier notation $a + b = b_1$ and $c = b_2$). Choose any $(1, 1)$-vertex y_1. Then, since $d(xy_1) \geqslant t$,

$$q(X) \geqslant q(x, X) \geqslant t - 1. \tag{52}$$

From the definition of a, b and c,

$$\begin{aligned} w(Y) &\geqslant (3a)/2 + 2(b + c) \\ &= 2(n - t - 3) - a/2. \end{aligned} \tag{53}$$

Suppose that $a \leqslant n - t - 4$. Then, from (42), (52) and (53), $q(G) \geqslant f^*(n, t)$. If $a = n - t - 3$ then $q(x, Y) = 0$ and, since $\delta(G) = t + 2$, $q(x, X) \geqslant t + 1$. Again using this inequality, (42) and (53), we have $q(G) \geqslant f^*(n, t)$. Hence if $\alpha(y) \geqslant 3$ for all $y \in \mathscr{B}_0$ we are done. □

So, finally, from Claim 3 we may suppose there exists a $(=2, =0)$-vertex $y \in Y$. Firstly, we show that all $(=2, =0)$-vertices have the same neighbours in X. Suppose that y_1, y_2 are $(=2, =0)$-vertices $(y_1 \neq y_2)$. Assume that y_1 and y_2 do not have the same neighbours in X. Then there exist distinct elements x_1, x_2, x_3 of X such that $q(x_1, y_1) = q(x_2, y_1) = q(x_3, y_2) = 0$. Hence,

$$\begin{aligned} q(X) &\geqslant q(x_1, X) + q(x_2, X) + q(x_3, X) - 3 \\ &\geqslant 3(t - 2) - 3. \end{aligned} \tag{54}$$

Always

$$w(Y) \geqslant n - t - 3. \tag{55}$$

So from (42), (54) and (55), $q(G) \geqslant f^*(n, t)$. We conclude that all $(=2, 0)$-vertices have the same neighbours in X. Set

$$A = \{y \in Y: \alpha(y) = 2, \beta(y) = 0\}, \quad |A| = a_1 \ (a_1 \geqslant 1).$$

Suppose that x and x^* are the distinct vertices of X such $q(\{x, x^*\}, A) = 0$. Set

$$a_2 = |\{y \in Y: (\alpha(y) \geqslant 3, \beta(y) = 0) \text{ or } (\alpha(y) = 1, \beta(y) = 1)\}|$$

$$a_3 = |\{y \in Y: \beta(y) = 2 \text{ or } (\beta(y) = 1 \text{ and } \alpha(y) \geqslant 2)\}|.$$

Now,

$$\begin{aligned} q(X) &\geqslant q(x, X) + q(x^*, X) \\ &\geqslant 2(t - 2) \end{aligned} \tag{56}$$

and

$$w(Y) \geqslant a_1 + \tfrac{3}{2}a_2 + 2a_3$$
$$= (n - t - 3) + \tfrac{1}{2}a_2 + a_3. \quad (57)$$

Then, from (42), (56) and (57), $q(G) \geqslant f^*(n, t)$ provided $\lceil a_2/2 \rceil + a_3 \geqslant 2$. So now assume that

$$\tfrac{1}{2}a_2 + a_3 \leqslant 1. \quad (58)$$

Suppose that $a_3 = 1$. Then, from (58), $a_2 = 0$. So if $y \in A$, $q(y, Y \backslash A) \leqslant 1$. Hence,

$$q(X) \geqslant q(x, X) + q(x^*, X)$$
$$\geqslant 2(t - 1) \quad (59)$$

and from (42), (57) and (59) we are done.

Finally, we may assume that $a_3 = 0$ and so, from (58), $a_2 \leqslant 2$, i.e. $|Y \backslash A| \leqslant 2$. Furthermore, if $y \in A$ and $q(y, Y \backslash A) \leqslant 1$ then again $q(X) \geqslant 2(t - 1)$ and from (42) and (57) we again deduce that $q(G) \geqslant f^*(n, t)$. Hence $|Y \backslash A| = 2$ and $q(A, Y \backslash A) = 2a_1$. Furthermore, $a_2 = |Y \backslash A| = 2$. Hence, from (58), equality holds in (57) or we are done. Therefore, $\alpha(y) \leqslant 3$ for all y in $Y \backslash A$. Hence, $q(A, Y \backslash A) \leqslant 6$. Hence, $a_1 \leqslant 3$ and $m = n - t - 3 \leqslant 5$ which is not the case. □

Theorem. *Suppose that $n \geqslant 2t + 2$ $(t \geqslant 17)$, then $f(n, t) = f^*(n, t)$.*

Proof. This follows from Lemmas 1, 5–7. □

Appendix

Proof of Lemma 4(ii). Recall that since $\bar{G}$ is connected $\beta_0 \leqslant s - 1$; this accounts for the increased complexity of the proof.

Throughout the proof we suppose that $y_0 \in Y$ and $\beta(y_0) = \beta_0$. Set $X_1 = N(y_0) \cap X$ and $X_0 = X \backslash X_1$. Notice that $|X_0| = s - \beta_0$. The proof is established by proving three claims. In Claims 1 and 2 we will assume (see Fig. 9) that there exists

$$y_1 \in Y,\ x_1 \in X_1 \quad \text{with } q(x_1, y_1) = 0. \quad (A.1)$$

Set $Y' = Y \backslash \{y_1\}$. Then $|Y'| = m - 1 (m^* = m - 1)$ and by Lemma 3(i) (using $\beta_0 \leqslant s - 1$)

$$\lambda(Y') \geqslant 2(m - 1)t + (m - 1)\lambda(m - 1, s - \beta_0, \beta_0). \quad (A.2)$$

The vertex y_1 contributes an additional (not counted by $\lambda(Y')$) number of edges towards the double-edge count $2q(G)$. Let $w(y_1)$ denote this additional contribution.

Claim A.1. $w(y_1) \geqslant 3t - s + 3\beta_0$.

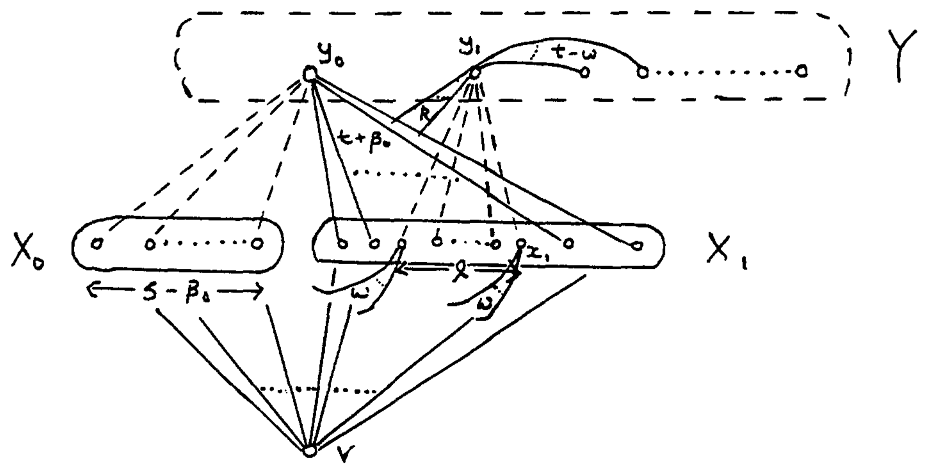


Fig. 9. Proof of Lemma 4(ii).

Proof (Claim A.1). Recall that by definition

$$q(y_1, X) = 2t + \beta(y_1). \tag{A.3}$$

Set $\alpha(y_1) = t - w$ $(t \geqslant w \geqslant 0)$, $k = q(y_1, X_0)$ and $l = |\{x \in X_1\colon q(x, y_1) = 0\}|$. Then, by definition,

$$k \leqslant s - \beta_0, \tag{A.4}$$

$$\beta_0 \leqslant \beta(y_1), \tag{A.5}$$

and the choice of x_1,

$$l = k - (\beta(y_1) - \beta_0) \geqslant 1. \tag{A.6}$$

If $w \leqslant k - 1$ then using (A.3)–(A.5),

$$\begin{aligned} w(y_1) &\geqslant 2t + (t - w) + 2\beta_0 \\ &\geqslant 3t - k + 1 + 2\beta_0 \\ &\geqslant 3t - s + 3\beta_0 + 1. \end{aligned} \tag{A.7}$$

Now suppose that $w \geqslant k$. Then for each of the l elements of X_1 such that $q(x, y_1) = 0$, $q(x, X) \geqslant w$ (since $d(xy_1) \geqslant t$). Hence from (A.3)–(A.6)

$$w(y_1) \geqslant 2t + (t - w) + 2\beta_0 + 2l(w - k) \tag{A.8}$$

$$\begin{aligned} &\geqslant 3t - k + 2\beta_0 \\ &\geqslant 3t - s + 3\beta_0. \end{aligned} \tag{A.9}$$

(Notice that in (A.8), since $w(y_1)$ is the *additional* contribution, we must deduct the $2lk$ edges already counted by $\lambda(Y')$). $\square$

Claim A.2. *Lemma* 3 *is true subject to the conditions of Claim A*.1.

Proof (Claim A.2). From (A.2) and Claim A.1,

$$\begin{aligned}2q(G) &\geqslant 2q(v,X)+\lambda(Y')+w(y_1)\\ &\geqslant 2(t+s)+2(m-1)t+(m-1)(s+\beta_0)\\ &\quad +2(s-\beta_0)(t-s+\beta_0)+(3t-s+3\beta_0)\\ &=G(\beta_0)\ \text{(say)}\quad (s-1\geqslant\beta_0\geqslant 0).\end{aligned}\tag{A.10}$$

$G(\beta_0)$ is minimized at one of its end-points $\beta_0=0$ or $\beta_0=s-1$:

$$\begin{aligned}G(0) &= 5t+s+(m-1)(2t+s)+2s(t-s)\\ &\geqslant 5t+2s-4+m(2s-1+2t)-\mathrm{pos}\langle 4s-2-(\theta+1)(s-1)\rangle\end{aligned}\tag{A.13}$$

if and only if

$$0\geqslant m(s-1)-2s(t-s)+2(t+s-2)-\mathrm{pos}\langle 4s-2-(\theta+1)(s-1)\rangle,$$

i.e.

$$\begin{aligned}0 &\geqslant (2(t-s)-(\theta+1))(s-1)-2s(t-s)+2(t+s-2)\\ &\quad -\mathrm{pos}\langle 4s-2-(\theta+1)(s-1)\rangle\end{aligned}\tag{A.14}$$

which is true. Furthermore,

$$G(s-1)=m(2s-1+2t)+5t+2s-4.\tag{A.15}$$

The claim is now a consequence of (A.10), (A.13) and (A.15). □

Because of Claim A.2 we may now assume that (A.1) does not hold, i.e.

$$\text{if } y\in Y \text{ then } q(y,X_1)=|X_1|\tag{A.16}$$

(see Fig. 10).

Set

$$F(\beta_0)=m(2t+s+\beta_0)+2(t-s+\beta_0)(s-\beta_0)+3t+2s.$$

Claim A.3. $2q(G)\geqslant F(\beta_0)$.

Proof (Claim A.3). Choose $w(\geqslant 0)$ so that

$$2q(G)=2q(v,X)+\lambda(Y)+w.\tag{A.17}$$

We show that $w\geqslant t$ as follows.

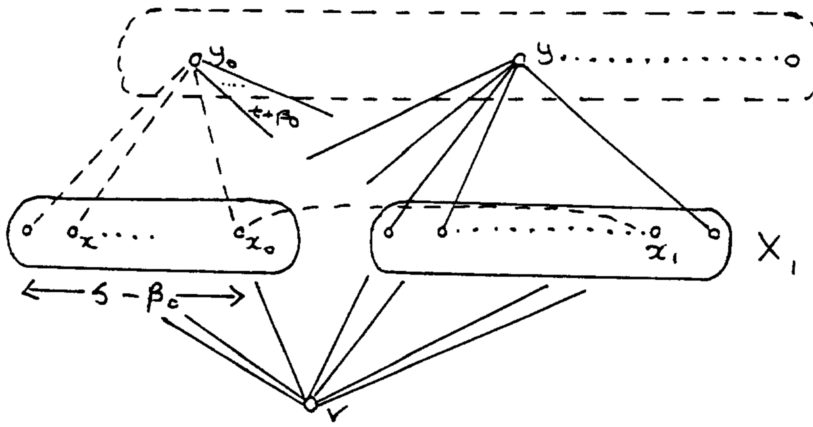


Fig. 10. Proof of Claim 2.

Using (A.16), since $\bar{G}$ is connected there exist $x_0 \in X_0$, $x_1 \in X_1$ with $q(x_0, x_1) = 0$. Suppose that $q(x_0, Y) \leqslant t - 2$. Then since $d(x_0 x_1) \geqslant t$

$$w \geqslant 2q(x_1, X) + q(x_0, Y)$$

$$\geqslant 2(t - 1 - q(x_0, Y)) + q(x_0, Y).$$

On the other hand, if $q(x_0, Y) \geqslant t$ then

$$w \geqslant q(x_0, Y) \geqslant t.$$

So, we may assume that $q(x_0, Y) = t - 1$ and hence that $w = t - 1$ or we are done.

Firstly, suppose now that $|X_0| \geqslant 2$, i.e. $s \geqslant \beta_0 + 2$. Take any $x \in X_0 (x \neq x_0)$. Then since $w = q(x_0, Y) = t - 1$, $q(x, Y) = 0$. Since $\delta(G) = t + s$, it follows that $q(x, X) \geqslant t + s - 1$. But since $w = t - 1$, using (A.17), $q(x, X) = t - s + \beta_0$ which is impossible. Hence $|X_0| = 1$, i.e. $\beta_0 = s - 1$. Again since $w = t - 1$, from (A.17), $q(x_0, X) = t - 1$ and $q(x_0, Y) = 0$. But $d(x_0 y_0) \geqslant t$ and hence, since $q(x_0, X) = t - 1$, $q(x_0, y) = 1$ for some y in Y. This contradiction proves that $w \geqslant t$.

From (A.17) and Lemma 3(i)

$$2q(G) \geqslant 2(t + s) + 2mt + m\lambda(m, s - \beta_0, \beta_0) + t$$

$$= F(\beta_0). \quad \square \tag{A.18}$$

Proof (Lemma 4(ii)). $F(\beta_0)$ has its minimum at its end-points, i.e. when $\beta_0 = 0$ or $\beta_0 = s - 1$. We have

$$F(0) = m(2t + s) + 2s(t - 2) + 3t + 2s \geqslant G(0) \tag{A.19}$$

which, from (A.13), gives the required inequality; furthermore,

$$F(s-1) = m(2s - 1 + 2t) + 5t + 2s - 2. \tag{A.20}$$

So from (A.19), (A.20) and Claim A.3 the proof is complete. □

References

[1] R. Gould, M. Jacobson, Neighborhood intersections and generalizations of Ore's Theorem. Graph theory, combinatorics, algorithms and applications, San Francisco, CA, 1989, SIAM, Philadelphia, PA, 1991, pp 198–206.
[2] E. Palmer, Graphical Evolution, Wiley, New York, 1985.
[3] C. Thomassen, 2-linked graphs, European J. Combin. 1 (1980) 371–378.